Physics 566, Quantum Optics
Problem Set #7
Due: Monday Nov. 15, 2010

Problem 1: Nonclassical light generation via the Kerr effect. (10 points)
In the classical (optical) Kerr effect, the index of refraction is proportional to the intensity. The quantum optical description is via the Hamiltonian,

$$\hat{H} = \frac{\hbar\chi^{(3)}}{2}\hat{a}^{\dagger 2}\hat{a}^2.$$

(a) Suppose we inject a strong coherent state into a nonlinear fiber with Kerr response. *Linearize* this Hamiltonian about the mean field via the substitution $\hat{a} = \alpha + \hat{b}$, and keep terms only up to quadratic order in $\hat{b}$ and $\hat{b}^\dagger$.
Show that the resulting Hamiltonian leads to squeezing.

(b) Now let's go beyond the linear approximation. Show that for a long time such that $\chi^{(3)}t = \pi$, the state becomes a Schrödinger cat, $\left(e^{i\pi/4}\left|-i\alpha\right\rangle + e^{-i\pi/4}\left|i\alpha\right\rangle\right)/\sqrt{2}$.

Note: Though in principle this is the solution, in practice this is not observed with light because losses and other noise sets in long before this kind of coherence can be established. More recently, the analogous experiment has been performed with Bose-Einstein condensates of atoms, which shows the same nonlinear dynamics (see Greiner *et al., Nature* **419**, 51-54 (5 September 2002)).

Problem 2: Twin beams and two-mode squeezed states. (15 points)
In lecture we discussed how parametric downconversion leads to correlated twin "signal" and "idler" beams as long as the phase matching conditions are satisfied,

$$\omega_p = \omega_s + \omega_i, \;\; \mathbf{k}_p = \mathbf{k}_s + \mathbf{k}_i.$$

Considering two nondegenerate modes, $\omega_\pm = \omega_p/2 \pm \Delta\omega$, the Hamiltonian is

$$\hat{H} = i\hbar G\left(\hat{a}_+^\dagger\hat{a}_-^\dagger e^{-i\phi} - \hat{a}_+\hat{a}_- e^{i\phi}\right),$$

where G is the coupling constant depending on the nonlinearity, pump amplitude, and vacuum mode strength. The state produced is known as a "two-mode squeezed vacuum state", $\hat{S}_{\pm}(\xi)|0\rangle_{+}\otimes|0\rangle_{-}=\exp\left[\xi\hat{a}_{+}\hat{a}_{-}-\xi^{*}\hat{a}_{+}^{\dagger}\hat{a}_{-}^{\dagger}\right]|0\rangle_{+}\otimes|0\rangle_{-}$, where $\xi=re^{i\phi}$ is the complex squeezing parameter for an interaction time t, $r=Gt$.

(a) Show that the generalized Bogoliubov transformations is

$$\hat{S}_{\pm}^{\dagger}(\xi)\hat{a}_{\pm}\hat{S}_{\pm}(\xi)=\cosh(r)\hat{a}_{\pm}-e^{-i\phi}\sinh(r)\hat{a}_{\mp}^{\dagger}.$$

(b) Show that the individual modes, $\hat{a}_{\pm}$, show no squeezing, but that squeezing exists in the *correlation* between the modes. Hint: consider quadratures,

$$\hat{X}_{\pm}(\theta)\equiv\frac{\hat{a}_{\pm}e^{i\theta}+\hat{a}_{\pm}^{\dagger}e^{-i\theta}}{2}\text{ and then }\hat{Y}(\theta,\theta')\equiv\left(\hat{X}_{+}(\theta)-\hat{X}_{-}(\theta')\right)/\sqrt{2}.$$

For the remaining parts, take ξ real.

(c) The two-mode squeezed state is an entangled state between the signal and idler as we know from the perturbative limit of twin photons. Show that in the Fock basis

$$\hat{S}_{\pm}(r)|0\rangle_{+}\otimes|0\rangle_{-}=(\cosh(r))^{-1}\sum_{n=0}^{\infty}(\tanh(r))^{n}|n\rangle_{+}\otimes|n\rangle_{-}.$$

Hint: Use the "disentangling theorem" (D. R. Traux, Phys. Rev. D **31**, 1988 (1985)):

$$e^{r\left(\hat{a}_{+}^{\dagger}\hat{a}_{-}^{\dagger}-\hat{a}_{+}\hat{a}_{-}\right)}=e^{\Gamma\hat{a}_{+}^{\dagger}\hat{a}_{-}^{\dagger}}e^{-g\left(\hat{a}_{+}^{\dagger}\hat{a}_{+}+\hat{a}_{-}^{\dagger}\hat{a}_{-}+1\right)}e^{-\Gamma\hat{a}_{+}\hat{a}_{-}}.$$

where $\Gamma=\tanh(r),\ g=\ln(\cosh(r))$

The photons are produced with perfect correlations between the modes. This is known as "number squeezing".

(d) Show that the marginal density operator for each mode is a *thermal state* with mean photon number $\bar{n}=\sinh^{2}(r)$.

(e) This entangled state produced in twin beam generation is very close to the form considered by Einstein-Podolfsky-Rosen in their famous paradox. **Show** that the Wigner function for our state in the two modes is,

$$W(\alpha_+,\alpha_-) = \frac{4}{\pi^2}\exp\left\{-e^{-2r}\left[(x_+ - x_-)^2 + (p_+ + p_-)^2\right] - e^{+2r}\left[(x_+ + x_-)^2 + (p_+ - p_-)^2\right]\right\}$$
$$\to C\delta(x_+ + x_-)\delta(p_+ - p_-)$$

where $\alpha_\pm = x_\pm + ip_\pm$ the final expression is the limit of infinite squeezing, $r \to \infty$. For mechanical degrees of freedom, $x_\pm$ and $p_\pm$ represent position and momentum of two particles which are perfectly correlated. The quantum optical implementation maps these onto mode quadratures which are very tightly correlated. Violations of Bell's inequalties can then be measured (see Z.Y. Ou *et al*., Phys. Rev Lett. **68**, 3663 (1992)).

(f) Extra credit: The Wigner function is positive, but clearly the EPR state clear has nonclassical features. Your thoughts?

Problem 3: Gaussian States in Quantum Optics (15 points)

The set of states whose quadrature fluctuations are Gaussian distributed about a mean value is an important class in quantum optics. A general squeezed state is such an example. In this problem, we explore Gaussian states, their relationship to squeezing, and the canonical algebra of phase space.

Consider a field of *n-modes*, with quadrature defined by an ordered vector:

$$\mathbf{Z} = (X_1, P_1, X_2, P_2, \ldots, X_n, P_n).$$

The operators associated with these quadratures satisfy a set of canonical commutators relations that can be written compactly as,

$$\left[\hat{Z}_i, \hat{Z}_j\right] = \frac{i}{2}\Sigma_{ij}, \text{ where } \Sigma_{ij} = \bigoplus_{i=1}^{n}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is a skew-symmetric matrix.}$$

We define an "inner product" in phase space as $\left(\mathbf{Z}\middle|\mathbf{Q}\right) = Z_i \Sigma_{ij} Q_j$ (summed over repeated indices through this problem).

(a) **Show that** the phase space displacement operator can be written

$$\hat{D}(\mathbf{Z}) = \exp\left\{-2i\left(\mathbf{Z}\middle|\hat{\mathbf{Z}}\right)\right\}$$

A *Gaussian state* is one whose Wigner function is a Gaussian function on phase space. Recall the characteristic function of a quantum state is defined $\chi(\mathbf{Z}) = Tr\left(\hat{\rho}\hat{D}(\mathbf{Z})\right)$. The general form of the characteristic function for a Gaussian state with is:

$$\chi(\mathbf{Z}) = \exp\left\{-2\left(\mathbf{Z}|\mathbf{C}|\mathbf{Z}\right) + 2i\left(\mathbf{d}|\mathbf{Z}\right)\right\}.$$

Where C_{ij} is known as the covariance matrix, and d_i is a real vector.

(b) **Show that:** $\left\langle \hat{Z}_i \right\rangle = d_i$, and $\frac{1}{2}\left\langle \Delta\hat{Z}_i\Delta\hat{Z}_j + \Delta\hat{Z}_j\Delta\hat{Z}_i \right\rangle = C_{ij}$, where $\Delta\hat{Z}_i \equiv \hat{Z}_i - \left\langle \hat{Z}_i \right\rangle$.
Hint: Recall how moments are found from the characteristic function.

The Gaussian state is thus determined by the mean position in phase space and the covariance of all the fluctuations.

(c) **Find** the Wigner function for a state with the general form of the characteristic function.

Let us restrict our attention to Gaussian states with zero mean (the mean is irrelevant to the statistics and can always be removed via a displacement operation). Consider now unitary transformations on the state. A particular class of transformations is the set that act as linear canonical transformations, i.e.

$$\hat{U}^\dagger \hat{Z}_i \hat{U} = S_{ij}\hat{Z}_j \text{, where } S_{ij} \text{ is a symplectic matrix, defined by } S^T \Sigma S = \Sigma .$$

A unitary map on the state transforms the state according to

$$\chi(\mathbf{Z}) \Rightarrow \chi'(\mathbf{Z}) = Tr\left(\hat{U}\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\right) = Tr\left(\hat{\rho}\hat{U}^\dagger \hat{D}(\mathbf{Z})\hat{U}\right).$$

(d) **Show that** for a symplectic transformation, the characteristic function transforms as

$$\chi(\mathbf{Z}) \Rightarrow \chi(\mathbf{SZ})$$

and thus the action of the unitary is to *preserve the Gaussian statistics, by* transforming covariance matrix as $\mathbf{C} \Rightarrow \mathbf{S}^T\mathbf{CS}$.

(e) Show that the following operations preserve Gaussian statistics:

- Linear optics: $\hat{U} = \exp\left(-i\theta_{ij}\hat{a}_i^\dagger \hat{a}_j\right)$
- Squeezing: $\hat{U} = \exp\left(\zeta_{ij}^* \hat{a}_i \hat{a}_j - \zeta_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger\right)$

(f) For each of these, show how the covariance matrix of the Gaussian transforms.

(g) Starting with the vacuum (a Gaussian state) we apply the squeezing operator above. Show that the symplectic transformation on the covariant matrix leads to the expected result.